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Centers and medians of distance-hereditary graphs

Hong-Gwa Yeh^a, Gerard J. Chang^{b,*}^a*Department of Mathematics, National Central University, Chung-Li 320, Taiwan*^b*Department of Mathematics, National Taiwan University, Taipei 106, Taiwan*

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Abstract

A graph is distance-hereditary if the distance between any two vertices in a connected induced subgraph is the same as in the original graph. In this paper, we study metric properties of distance-hereditary graphs. In particular, we determine the structures of centers and medians of distance-hereditary and related graphs. The relations between eccentricity, radius, and diameter of such graphs are also investigated.

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1. Introduction

This paper investigates metric properties of centers and medians of distance-hereditary graphs.

Suppose $G = (V, E)$ is a graph with vertex set V and edge set E . The *distance* $d_G(x, y)$ or $d(x, y)$ between two vertices x and y in the graph G is the minimum number of edges of an x – y path in G . The *eccentricity* $e_G(v)$ of a vertex v in G is $\max_{x \in V} d(v, x)$. The *diameter* $\text{diam}(G)$ of G is the largest eccentricity of a vertex in G , and the *radius* $\text{rad}(G)$ is the smallest. The *center* of G is the set

$$C(G) = \{v \in V : e_G(v) \leq e_G(x) \text{ for all } x \in V\}.$$

* Corresponding author.

E-mail addresses: hgyeh@math.ncu.edu.tw (H.-G. Yeh), gjchang@math.ntu.edu.tw (G.J. Chang).

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The *distance sum* $D_G(v)$ of v is $\sum_{x \in V} d_G(v, x)$. The *median* of G is the set

$$M(G) = \{v \in V: D_G(v) \leq D_G(x) \text{ for all } x \in V\}.$$

Suppose w is a real-valued function on V . The *w-distance sum* $D_{G,w}(v)$ of v is $\sum_{x \in V} d_G(v, x)w(x)$. The *w-median* of G is the set

$$M_w(G) = \{v \in V: D_{G,w}(v) \leq D_{G,w}(x) \text{ for all } x \in V\}.$$

The *local w-median* of G is the set

$$LM_w(G) = \{v \in V: D_{G,w}(v) \leq D_{G,w}(x) \text{ for all } x \text{ adjacent to } v\}.$$

We very often also call the subgraph induced by the center (respectively, median, w -median, local w -median) simply the center (respectively, median, w -median, local w -median) of the graph.

The problem of determining shapes of centers and medians for different classes of graphs have been extensively studied in the literature, see Refs. [1,3–5,6,10,11,14,15,18–42]. The earliest such kind of result due to Jordan [18] is that the center or the median of a tree is either a single vertex or two adjacent vertices. Slater [30,31] examined the structure of variations of centers for trees. Proskurowski [26] proved that the center of a maximal outplanar graph is one of seven special graphs. As a generalization, he [27] found all possible centers of 2-trees and showed that the center of a 2-tree is biconnected. Laskar and Shier [19] proved that the center of a connected chordal graph is connected. Chang [5] showed that the center of a connected chordal graph is distance invariant and biconnected. He also gave a characterization of a biconnected chordal graph of diameter 2 and radius 1 to be the center of some chordal graph. Yushmanov [38,40] showed that the center of a connected chordal graph is m -convex and is either a complete graph or its connectivity is no smaller than the connectivity of the block in which it lies. Soltan and Chepoi [36] proved that the center of a connected chordal graph has diameter at most 3.

Slater [31] studied the structure of various types of medians for trees. He [32] also proved that for every graph H there exists a graph G whose median is H , and that the median of a 2-tree is isomorphic to K_1 , K_2 or K_3 . Yushmanov [41] and Nieminen [25] showed that the median of a Ptolemaic graph is a complete graph. Lee and Chang [20] proved that the w -median of a connected strongly chordal graph is a complete graph when the function w is positive. Wittenberg [37] proved that for any chordal graph the local w -median coincides with the w -median if a certain neighborhood condition holds.

Many results on centers and medians are on graphs with tree-like structures. In this paper, we focus on distance-hereditary graphs, which include trees and cographs. The paper is organized as follows. In Section 2, we survey some properties of distance-hereditary graphs. We also introduce the concept of concavity of a path, and derive two lemmas that are useful in this paper. Section 3 shows that the center of a distance-hereditary graph (respectively, bipartite distance-hereditary graph) is either a connected graph with diameter at most 3 or a cograph (respectively, an independent set of G). Section 4 shows that the w -median of a distance-hereditary graph with a positive

function w is a cograph, and the w -median of a bipartite distance-hereditary graph G with a positive function w is either a connected cograph or an independent set of G . In Section 5, we show that the w -median of a distance-hereditary graph with a positive function w “nearly” coincides with its local w -median.

2. Preliminaries of distance-hereditary graphs

This section gives a brief introduction to distance hereditary and related graphs. We also introduce the concept of concavity of a path, and give two related lemmas that are useful in the paper.

Suppose S is a vertex subset of a graph $G=(V,E)$. Denote $G[S]$ the subgraph of G induced by S . The *deletion* of S from G , denoted by $G-S$, is the induced subgraph $G[V-S]$. The *neighborhood* $N_G(v)$ or $N(v)$ of a vertex v is the set of all vertices adjacent to v . An *induced* (or *chordless*) *path* is a path v_1, v_2, \dots, v_n in which v_i is not adjacent to v_j whenever $|i-j| \neq 1$. A vertex subset S of a graph G is called *m-convex* if all the vertices of all the induced path joining vertices of S lie in S . Two vertices x and y are *connected* if there is an x - y path in G ; otherwise they are *disconnected*. The *distance* between two vertex subsets A and B is $d_G(A,B) = \min\{d_G(a,b) : a \in A \text{ and } b \in B\}$.

The *hanging* h_u of a connected graph $G=(V,E)$ at a vertex $u \in V$ is the collection of sets $L_0(u), L_1(u), \dots, L_t(u)$ (or L_0, L_1, \dots, L_t if there is no ambiguity), where $t = \max_{v \in V} d_G(u,v)$ and $L_i(u) = \{v \in V : d_G(u,v) = i\}$ for $0 \leq i \leq t$. $L_i(u)$ is called the *level* i of the hanging h_u . For any $1 \leq i \leq t$ and any vertex $v \in L_i$, let $N'(v) = N(v) \cap L_{i-1}$.

A graph G is *distance hereditary* if each connected induced subgraph F of G has the property that $d_F(u,v) = d_G(u,v)$ for every pair of vertices u and v in F . Distance-hereditary graphs were introduced by Howorka [16]. The characterizations and recognitions of distance-hereditary graphs have been studied in [2,8,13,16]. A graph is *chordal* if every cycle of length greater than three has a chord. A *cograph* is a graph containing no induced path of four vertices, see [7]. A graph is *Ptolemaic* if for any four vertices x, y, z, w of it, the Ptolemy inequality $d(x,y)d(z,w) \leq d(x,z)d(y,w) + d(x,w)d(y,z)$ holds. It was shown in [17] that G is Ptolemaic if and only if G is distance hereditary and chordal. Some containment relationships among these and other families of graphs are as follows:

trees \subset block graphs \subset Ptolemaic graphs \subset distance-hereditary graphs,
 trees \subset bipartite distance-hereditary graphs \subset distance-hereditary graphs,
 cographs \subset distance-hereditary graphs,
 Ptolemaic graphs \subset strongly chordal graphs \subset chordal graphs.

Theorem 1 (Bandelt and Mulder [2], D’Atri and Moscarini [8], Hammer and Maffray [13], Howorka [16]). *For any connected graph $G=(V,E)$ the following statements are equivalent:*

- (1) G is a distance-hereditary graph.
- (2) Every cycle of length at least five in G has two crossing chords.

- (3) Every induced path in G is a shortest path.
 (4) For every hanging $h_u = (L_0, L_1, \dots, L_t)$ of G and every pair of vertices $x, y \in L_i$ ($1 \leq i \leq t$) that are in the same component of $G - L_{i-1}$, $N'(x) = N'(y)$.

Theorem 2 (Bandelt and Mulder [2], D'Atri and Moscarini [8], Hammer and Maffray [13], Howorka [16]). Suppose $h_u = (L_0, L_1, \dots, L_t)$ is the hanging of a connected distance-hereditary graph G at u . Then, every L_i induces a cograph. Moreover, if G is bipartite, then every L_i is an independent set of G .

Suppose $G = (V, E)$ is a graph. For two vertices x and y in V , let

$$V_{xy} = \{v \in V : d_G(x, v) < d_G(y, v)\}.$$

Define the function ℓ_u from V to non-negative integers by: $\ell_u(v) = k$ whenever $d_G(u, v) = k$, or equivalently, v is in L_k for the hanging $h_u = (L_0, L_1, \dots, L_t)$ of G at u . A path $P: x_0, x_1, \dots, x_k$ is *u-concave downward* if $\ell_u(x_0) > \ell_u(x_1) > \dots > \ell_u(x_{r-1}) > \ell_u(x_r) = \ell_u(x_{r+1}) = \dots = \ell_u(x_{r'}) < \ell_u(x_{r'+1}) < \dots < \ell_u(x_{k-1}) < \ell_u(x_k)$ for some $0 \leq r \leq r' \leq k$ with $0 \leq r' - r \leq 1$. P is *monotone* if $r = k$ or $r' = 0$.

The following two lemmas are useful in this paper. Note that they are trivial for the case when the distance-hereditary graphs are trees.

Lemma 3. Suppose $h_u = (L_0, L_1, \dots, L_t)$ is the hanging of a connected distance-hereditary graph G at u . For any two vertices x and y in G , there exists a shortest x - y path which is *u-concave downward*.

Proof. Suppose $P: x = x_0, x_1, \dots, x_k = y$ is a shortest x - y path, where $d_G(x, y) = k$. We may assume that P is chosen such that $s(P) = \sum_{j=0}^k \ell_u(x_j)$ is as small as possible. Let i be the largest index such that x_0, x_1, \dots, x_i is *u-concave downward*. Note that $i \geq 1$. In fact $i = k$ and so the lemma holds. Suppose to the contrary that $i < k$. Then one of the following cases holds:

- (1) $\ell_u(x_{i-1}) = \ell_u(x_i) > \ell_u(x_{i+1})$.
- (2) $\ell_u(x_{i-1}) < \ell_u(x_i) = \ell_u(x_{i+1})$.
- (3) $\ell_u(x_{i-1}) = \ell_u(x_i) = \ell_u(x_{i+1})$.
- (4) $\ell_u(x_{i-1}) < \ell_u(x_i) > \ell_u(x_{i+1})$.

For case (1) or (2), by Theorem 1 (4), x_{i-1} is adjacent to x_{i+1} , a contradiction to that P is a shortest path. For case (3) or (4), by Theorem 1 (4), x_{i-1} and x_{i+1} have a common neighbor x'_i with $\ell_u(x'_i) = \ell_u(x_{i-1}) - 1$. Then, the path P' resulting from P by replacing x_i with x'_i is also a shortest x - y path whose $s(P') < s(P)$, a contradiction to the choice of P . This completes the proof of the lemma. \square

Lemma 4. Suppose $P: x_0, x_1, \dots, x_k$ is an induced path of a distance-hereditary graph $G = (V, E)$. If $k = 2$ with G chordal or $k \geq 3$, then $V_{x_0x_1}$ is a proper subset of $V_{x_{k-1}x_k}$.

Proof. Suppose $h_{x_0} = (L_0, L_1, \dots, L_t)$ is the hanging of G at x_0 . Assume that there exists some vertex $v \in V_{x_0x_1} - V_{x_{k-1}x_k}$. By Lemma 3, there exists an x_0 -concave downward

shortest v - x_k path $P' : v = y_0, y_1, \dots, y_r, \dots, y_{r'}, \dots, y_{k'} = x_k$, where y_r (respectively, $y_{r'}$) is the first (respectively, last) vertex of P' in a smallest level L_f with $0 \leq r' - r \leq 1$.

For the case when $f \geq 2$, y_r and x_f are in the same component of $G - L_{f-1}$. According to Theorem 1 (4), y_r is adjacent to x_{f-1} and so $v = y_0, y_1, \dots, y_r, x_{f-1}, x_{f-2}, \dots, x_1, x_0$ is a shortest v - x_0 path, contradicting that $v \in V_{x_0x_1}$.

For the case when $f = 0$, $v = y_0, y_1, \dots, y_r, x_1, x_2, \dots, x_{k-1}, x_k$ is a shortest v - x_k path, contradicting that $v \notin V_{x_{k-1}x_k}$.

Now, suppose $f = 1$. For the case when $k \geq 3$, $y_{r'+1}$ and x_2 are in the same component of $G - L_1$. According to Theorem 1 (4), $y_{r'}$ is adjacent to x_2 and so $v = y_0, y_1, \dots, y_{r'}, x_2, x_3, \dots, x_{k-1}, x_k$ is a shortest v - x_k , contradicting that $v \notin V_{x_{k-1}x_k}$. For the case when $k = 2$ and G is chordal, $y_{r'}$ and x_1 adjacent to x_2 imply that they are also adjacent to x_0 , i.e., $y_{r'}x_0x_1x_2y_{r'}$ is a cycle of G . By the definition of a chordal graph, $y_{r'}$ is adjacent to x_1 . If $r = r'$, then $d_G(v, x_1) \leq d_G(v, x_0)$, contradicting that $v \in V_{x_0x_1}$. If $r = r' - 1$, then $x_0x_1x_2y_{r'}y_{r'}x_0$ is a cycle of G . By Theorem 1 (2), y_r is adjacent to x_1 , and hence $d_G(v, x_1) < d_G(v, x_2)$ contradicting that $v \notin V_{x_{k-1}x_k}$.

Therefore, $V_{x_0x_1}$ is a subset of $V_{x_{k-1}x_k}$; and in fact a proper subset as $x_{k-1} \in V_{x_{k-1}x_k} - V_{x_0x_1}$. This completes the proof of the lemma. \square

3. Centers

The purpose of this section is to investigate the shapes of centers of distance-hereditary graphs. We in fact study centers in a more general setting as follows. Suppose S is a non-empty subset of V in a graph $G = (V, E)$. The S -eccentricity $e_{G,S}(v)$ of a vertex v in G is $\max_{x \in S} d(v, x)$.

The S -center of G is $C_S(G) = \{v \in V : e_{G,S}(v) \leq e_{G,S}(x) \text{ for all } x \in V\}$.

The anticenter of G is $AC(G) = \{v \in V : e_G(v) \geq e_G(x) \text{ for all } x \in V\}$.

The S -anticenter of G is $AC_S(G) = \{v \in V : e_{G,S}(v) \geq e_{G,S}(x) \text{ for all } x \in V\}$.

Theorem 5. Suppose S is a non-empty vertex set of a distance-hereditary graph $G = (V, E)$. If H is a connected component of $G[T]$, where $T \subseteq V$ with $e_{G,S}(x) = e_{G,S}(y)$ for every two vertices x and y in T , then $\text{diam}(H) \leq 3$. If moreover G is Ptolemaic, then $\text{diam}(H) \leq 2$.

Proof. Suppose x and y are two vertices in $V(H)$ such that $d_H(x, y) = \text{diam}(H) = k$. Choose an induced x - y path $P : x = x_0, x_1, \dots, x_k = y$ in H . Note that P is also an induced path of G and $e_{G,S}(x_0) = e_{G,S}(x_1) = \dots = e_{G,S}(x_k)$. Let $e_{G,S}(x_1) = d_G(x_1, z)$ for some vertex $z \in S$. Then, for $0 \leq i \leq k$, we have

$$d_G(x_i, z) \leq e_{G,S}(x_i) = e_{G,S}(x_1) = d_G(x_1, z), \text{ i.e., } z \in V_{x_i x_1} \text{ or } d_G(x_i, z) = d_G(x_1, z).$$

We first prove that $k \leq 3$. Suppose to the contrary that $k \geq 4$. If $z \in V_{x_0 x_1}$, then $z \in V_{x_2 x_3}$ and $z \in V_{x_3 x_4}$ by Lemma 4. These imply that $e_{G,S}(x_4) \geq d_G(x_4, z) = d_G(x_3, z) + 1 = d_G(x_2, z) + 2 > d_G(x_1, z) = e_{G,S}(x_1)$, a contradiction. Hence $d_G(x_0, z) = d_G(x_1, z)$. We then hang G at z . Note that x_0 and x_1 are in the same level. If x_2 is in level $d_G(x_2, z) = d_G(x_1, z) - 1$, then x_0 is adjacent to x_2 by Theorem 1 (4), a contraction. Hence

$d_G(x_1, z) \leq d_G(x_2, z)$. Since $d_G(x_0, z) = d_G(x_1, z)$, we have $z \notin V_{x_1x_0}$. By Lemma 4, $z \notin V_{x_3x_2}$ and so $d_G(x_2, z) \leq d_G(x_3, z)$. Therefore, $d_G(x_0, z) = d_G(x_1, z) \leq d_G(x_2, z) \leq d_G(x_3, z) \leq d_G(x_1, z)$ and so $d_G(x_0, z) = d_G(x_1, z) = d_G(x_2, z) = d_G(x_3, z)$. Now, consider the hanging of graph G at z . Then x_0, x_1, x_2, x_3 are in the same level. By Theorem 1 (4), there exists a vertex z^* adjacent to x_0, x_1, x_2 , and x_3 . Note that $z^*x_0x_1x_2x_3z^*$ is a cycle of length 5 without crossing chords, a contradiction. Hence $k \leq 3$, i.e., $\text{diam}(H) \leq 3$.

Next, we prove that $k \leq 2$ when G is Ptolemaic. Suppose to the contrary that $k \geq 3$. If $z \in V_{x_0x_1}$, then $z \in V_{x_1x_2}$ by Lemma 4. This implies that $e_{G,S}(x_2) \geq d_G(x_2, z) > d_G(x_1, z) = e_{G,S}(x_1)$, a contradiction. Hence $z \notin V_{x_0x_1}$ and so $d_G(x_0, z) = d_G(x_1, z)$ and $z \notin V_{x_1x_0}$. Then, by Lemma 4, $z \notin V_{x_2x_1}$ and $z \notin V_{x_3x_2}$. Therefore, $d_G(x_0, z) = d_G(x_1, z) \leq d_G(x_2, z) \leq d_G(x_3, z) \leq d_G(x_1, z)$ and so $d_G(x_0, z) = d_G(x_1, z) = d_G(x_2, z) = d_G(x_3, z)$. Again, we can get a cycle $z^*x_0x_1x_2x_3z^*$ of length 5 without crossing chords, a contradiction. Hence $k \leq 2$, i.e., $\text{diam}(H) \leq 2$. \square

Corollary 6. Suppose S is a non-empty vertex set in a distance-hereditary graph G . If H is a connected component of the subgraph induced by $C_S(G)$ or $AC_S(G)$, then $\text{diam}(H) \leq 3$. If moreover G is Ptolemaic, then $\text{diam}(H) \leq 2$.

The distance-hereditary graphs G_1 and G_2 and the Ptolemaic graphs G_3 and G_4 in Fig. 1 show that the bounds in Corollary 6 are sharp. Note that $C(G_1) = \{a_1, b_1, c_1, d_1, e_1, f_1\}$, $AC(G_2)$ has a connected component $G_2[\{a_2, b_2, c_2, d_2\}]$, $C(G_3) = \{a_3, b_3, c_3, d_3, e_3\}$, and $AC(G_4)$ has a connected component $G_4[\{a_4, b_4, c_4, d_4, e_4\}]$.

Because distance-hereditary graphs have a “tree like” structure of adjacency, one may expect that their centers are “small” and “compact”. The following lemma supports such expectations.

Lemma 7. If V_1 and V_2 are the vertex sets of two distinct components of the S -center of a distance-hereditary graph $G = (V, E)$, then $d_G(V_1, V_2) = 2$.

Proof. Assume that $d_G(V_1, V_2) \geq 3$. Then there exists an induced path $P: x_0, x_1, \dots, x_k$, where $k \geq 3$, $x_0 \in V_1$, $x_k \in V_2$, but $x_1, x_{k-1} \notin C_S(G)$. Assume z is a vertex in S with $d_G(x_{k-1}, z) = e_{G,S}(x_{k-1})$. Since $x_k \in C_S(G)$ and $x_{k-1} \notin C_S(G)$, $d_G(x_k, z) \leq e_{G,S}(x_k) < e_{G,S}(x_{k-1}) = d_G(x_{k-1}, z)$ and so $z \in V_{x_kx_{k-1}}$. By Lemma 4, we then have $z \in V_{x_1x_0}$. Let $h_z = (L_0, L_1, \dots, L_t)$ be the hanging of G at z . Since $x_0 \in C_S(G)$ and $x_{k-1} \notin C_S(G)$, we have $d_G(x_0, z) \leq e_{G,S}(x_0) < e_{G,S}(x_{k-1}) = d_G(x_{k-1}, z)$. Therefore, the relative positions of x_0, x_1, x_{k-1}, x_k are as shown in Fig. 2(a). Thus, there exists a vertex x_j in the path P such that x_j and x_k are in the same level, say L_i , of h_z , and $x_jx_{j+1}x_{j+2} \dots x_k$ is a path in $G - L_{i-1}$ (see Fig. 2(b)). Then, by Theorem 1 (4), x_k is adjacent to x_{j-1} , contrary to that P is an induced path. Therefore, $d_G(V_1, V_2) \leq 2$. \square

Theorem 8. If G is a distance-hereditary graph, then the S -center $C_S(G)$ is either a connected graph of diameter 3 or a cograph. If moreover G is a bipartite distance-hereditary graph, then the S -center $C_S(G)$ is either a connected graph of diameter ≤ 3 or an independent set of G .

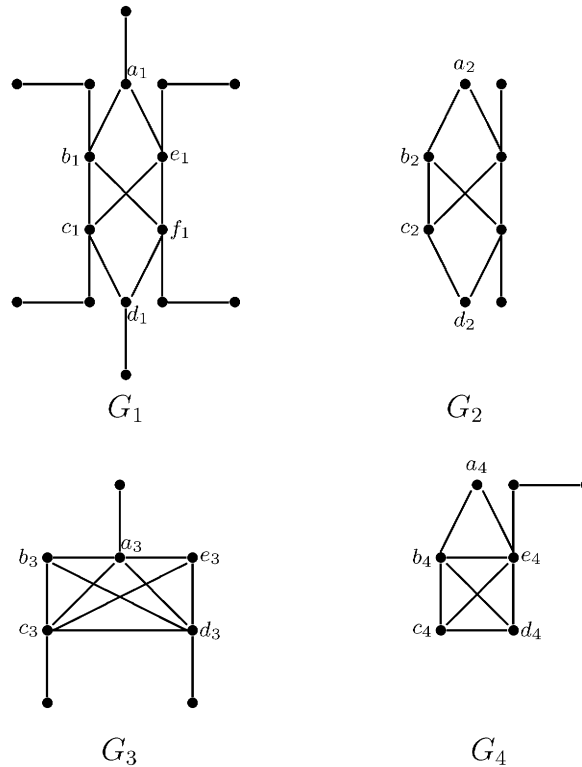


Fig. 1. Examples for which the bounds in Corollary 6 are sharp.

Proof. First, if $H = G[C_S(G)]$ is connected, then the theorem follows immediately from Theorem 5. Hence, we may assume that H is disconnected. Choose any two distinct components H_1 and H_2 . Then, by Lemma 7, there exists an induced path xzy in G such that $x \in V(H_1)$, $y \in V(H_2)$ and $z \notin C_S(G)$. Suppose w is a vertex in S with $d_G(w, z) = e_{G,S}(z)$. Then $d_G(w, x) \leq e_{G,S}(x) < e_{G,S}(z) = d_G(w, z)$ and so $d_G(w, z) = d_G(w, x) + 1$. Similarly, $d_G(w, z) = d_G(w, y) + 1$.

Let $h_w = (L_0, L_1, \dots, L_t)$ be the hanging of G at w . Note that x and y lie on the same level L_i of h_w and $z \in L_{i+1}$. Now for every vertex $x' \in H_1$, $d_G(w, x') \leq e_{G,S}(x') < e_{G,S}(z) = d_G(w, z)$ and so $x' \in L_r$ for some $r \leq i$. In fact $r = i$. Suppose to the contrary that $x' \in L_r$ for some $r \leq i - 1$. Then H_1 has an x - x' path P laying above level $i + 1$. Hence P contains an edge uv such that y and u are in the same component of $G - L_{i-1}$ and $v \in L_{i-1}$. Thus, by Theorem 1 (4), y is adjacent to v and hence $y \in H_1$, a contradiction. Therefore, every vertex of H_1 lies on level L_i . Hence H_1 is a cograph by Theorem 2. Moreover, if G is also bipartite, then L_i is an independent set of G by Theorem 2, and so is H_1 . This completes the proof of the theorem. \square

As described in [5], Hedetniemi proved that any graph H is isomorphic to the center of some graph G of diameter 4 and radius 2. When H is a cograph, an analogous

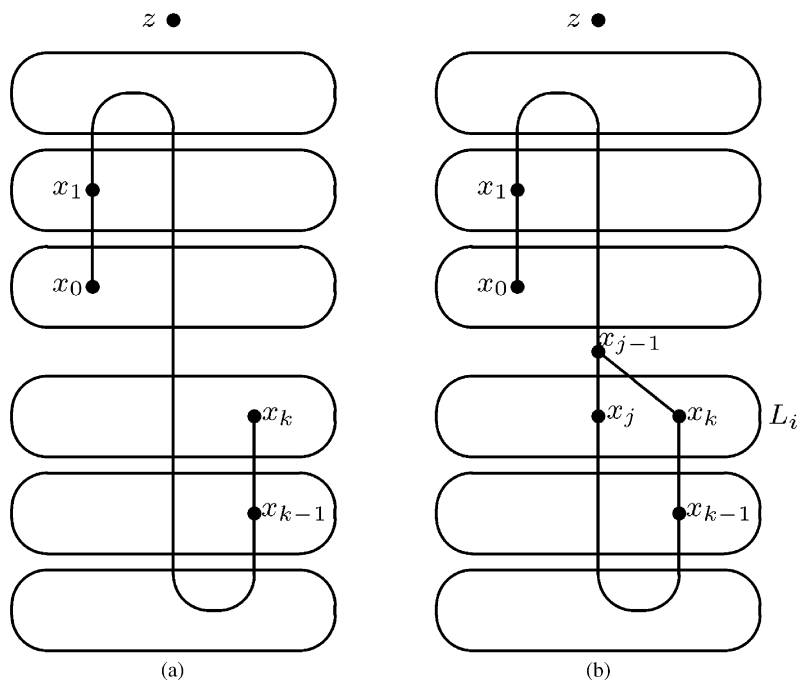


Fig. 2. For the proof of Lemma 7.

result for distance-hereditary graph G is the following theorem. Using the proof of this theorem, we can construct a distance-hereditary graph whose center induces a graph with arbitrary number of components. Also the center of a bipartite distance-hereditary graph can be an independent set of arbitrarily large size.

Theorem 9. *For any given cograph H there exists a connected distance-hereditary graph G whose center is isomorphic to H .*

Proof. We construct G by adding four new vertices u, v, w, x into H such that v and w are adjacent to all vertices of H , u is adjacent only to v and x only to w . It is clear that G is a distance-hereditary graph whose center is isomorphic to H . \square

4. Medians

This section discusses the structures of medians of distance-hereditary graphs. We again study medians in a more general setting. Suppose S is a non-empty subset of V . The S -distance sum $D_{G,S}(v)$ is equal to $\sum_{x \in S} d_G(v, x)$. The S - w -distance sum $D_{G,S,w}(v)$ of v is $\sum_{x \in S} d_G(v, x)w(x)$. The S -median (also called the S -centroid [31]) of G is the set

$$M_S(G) = \{v \in V : D_{G,S}(v) \leq D_{G,S}(x) \text{ for all } x \in V\}.$$

The S - w -median of G is the set

$$M_{S,w}(G) = \{v \in V : D_{G,S,w}(v) \leq D_{G,S,w}(x) \text{ for all } x \in V\}.$$

The *antimedian* of G is the set

$$AM(G) = \{v \in V : D_G(v) \geq D_G(x) \text{ for all } x \in V\}.$$

Lemma 10 (Entringer et al. [10], Slater [31]). *If a and b are two adjacent vertices of a graph $G = (V, E)$ with a function w on V , then $D_{G,w}(a) - D_{G,w}(b) = w(V_{ba}) - w(V_{ab})$.*

Proof. The lemma follows immediately from the fact that

$$\begin{aligned} D_{G,w}(a) - D_{G,w}(b) &= \sum_{x \in V_{ab}} \{d_G(x, a) - d_G(x, b)\} w(x) \\ &\quad - \sum_{x \in V_{ba}} \{d_G(x, a) - d_G(x, b)\} w(x). \quad \square \end{aligned}$$

Lemma 11. *Suppose $P : x_0, x_1, \dots, x_k$ is an induced path of a distance-hereditary graph $G = (V, E)$ with a function $w > 0$ (respectively, $w \geq 0$) on V . If either $k \geq 2$ with G chordal or $k \geq 3$, then $D_{G,w}(x_0) - D_{G,w}(x_1) > (\text{respectively, } \geq) D_{G,w}(x_{k-1}) - D_{G,w}(x_k)$.*

Proof. The lemma follows immediately from Lemmas 4 and 10. \square

Theorem 12. *For any $S \subseteq V$ of a Ptolemaic graph $G = (V, E)$ with a function $w \geq 0$, the S - w -median $M_{S,w}(G)$ is m -convex.*

Proof. Assume $M_{S,w}(G)$ is not m -convex. Then there exists an induced path $P : x_0, x_1, \dots, x_k$ in G with $k \geq 2$ such that $x_0, x_k \in M_{S,w}(G)$ but $x_1, x_{k-1} \notin M_{S,w}(G)$. Then, by Lemmas 10 and 4, we have

$$\begin{aligned} 0 &> D_{G,S,w}(x_0) - D_{G,S,w}(x_1) \\ &= w(V_{x_1x_0} \cap S) - w(V_{x_0x_1} \cap S) \\ &\geq w(V_{x_kx_{k-1}} \cap S) - w(V_{x_{k-1}x_k} \cap S) \\ &= D_{G,S,w}(x_{k-1}) - D_{G,S,w}(x_k) > 0, \end{aligned}$$

a contradiction. So, $M_{S,w}(G)$ is m -convex. \square

Corollary 13 (Soltan [35]). *The median of a Ptolemaic graph is connected.*

Corollary 14 (Slater [31]). *For any subset S of the vertices of a tree T , the S -median of T is connected.*

Theorem 15. *Suppose $G = (V, E)$ is a distance-hereditary graph with a function $w > 0$. If H is a connected component of $G[T]$, where $T \subseteq V$ having $D_{G,w}(x) = D_{G,w}(y)$ for*

every two vertices x and y in T , then H is a cograph. If moreover G is Ptolemaic, then H is a clique.

Proof. Suppose $P: x_0, x_1, \dots, x_k$ is an induced path in H . Note that P is also an induced path of G . If G is distance hereditary (respectively, Ptolemaic) and $k \geq 3$ (respectively, $k \geq 2$), then by Lemma 11 and the fact that $w > 0$, we have $D_{G,w}(x_0) - D_{G,w}(x_1) > D_{G,w}(x_{k-1}) - D_{G,w}(x_k)$, contrary to $D_{G,w}(x_0) = D_{G,w}(x_1) = D_{G,w}(x_{k-1}) = D_{G,w}(x_k)$. So, $k \leq 2$ (respectively, ≤ 1) and hence H is a cograph (respectively, a clique). \square

Corollary 16 (Nieminen [25], Yushmanov [38]). *The median of a Ptolemaic graph is a clique.*

Corollary 17 (Yushmanov [38]). *Every connected component of the subgraph induced by the antimedian of a Ptolemaic graph is a clique.*

It is worth pointing out that the S -median of a Ptolemaic graph does not have a theorem like Theorem 15. As shown in [31], there exist trees whose S -median contains a path of n vertices for any n . The following theorem shows that the median of a distance-hereditary graph nearly coincides with its local median.

Theorem 18. *Suppose G is a distance-hereditary graph with a function $w > 0$. If $x \in M_w(G)$ and $y \in LM_w(G)$, then $d(x, y) \leq 2$.*

Proof. Suppose $P: x = x_0, x_1, \dots, x_k = y$ is an induced x - y path of G . Assume $k \geq 3$. By Lemma 11, we have $0 \geq D_{G,w}(x_0) - D_{G,w}(x_1) > D_{G,w}(x_{k-1}) - D_{G,w}(x_k) \geq 0$, a contradiction. So $d_G(x, y) \leq 2$. \square

Theorem 19. *If G is a Ptolemaic graph with a function $w \geq 0$, then $M_w(G) = LM_w(G)$.*

Proof. Assume that $LM_w(G) - M_w(G) \neq \emptyset$. Pick $y \in LM_w(G) - M_w(G)$ and $x \in M_w(G)$ such that $d_G(x, y) = d_G(LM_w(G) - M_w(G), M_w(G))$. Suppose $P: x = x_0, x_1, \dots, x_k = y$ is an induced x - y path of G . Note that $k \geq 2$ and $x_1 \notin M_w(G)$. Hence, by Lemma 11, $0 > D_{G,w}(x_0) - D_{G,w}(x_1) \geq D_{G,w}(x_{k-1}) - D_{G,w}(x_k) \geq 0$, a contradiction. Therefore, $LM_w(G) - M_w(G) = \emptyset$ and so $M_w(G) = LM_w(G)$. \square

Theorem 20. *If $G = (V, E)$ is a distance-hereditary graph with a function $w > 0$, then its w -median is a cograph. If moreover G is bipartite distance hereditary, then its w -median is either a connected cograph or an independent set of G .*

Proof. The first part of the theorem follows from Theorem 15 immediately. To prove the second part, suppose V_1 and V_2 are the vertex sets of two distinct components of the w -median of G . For any two vertices $x \in V_1$ and $y \in V_2$, by Theorem 18, we have $d_G(x, y) = 2$. Now consider the hanging $h_x = (L_0, L_1, \dots, L_t)$ of G at x . Clearly $V_2 \subseteq L_2$ and hence V_2 is an independent set of G by Theorem 2. \square

5. Convexity and diameters

This section investigates metric properties for chordal graphs and distance-hereditary graphs.

A vertex subset S is called an x – y separator of G if x and y are in different components of $G - S$. An x – y separator S is said to be *minimal* if no proper subset of S is an x – y separator of G .

Theorem 21 (Dirac [9]). *Every minimal x – y separator of a chordal graph is a clique.*

Theorem 22. *If $G = (V, E)$ is a chordal graph and $S \subseteq V$, then the S -center $C_S(G)$ of G is m -convex.*

Proof. Assume to the contrary that $C_S(G)$ is not m -convex. Then there exist x and y in $C_S(G)$ with an induced x – y path P of G such that $|V(P)| \geq 3$ and $P \cap C_S(G) = \{x, y\}$. Suppose C is a minimal x – y separator of G . So, there exists a vertex $r \in P \cap C$ and hence $r \in C - C_S(G)$. Suppose $s' \in S$ with $d_G(s', r) = e_{G,S}(r)$. If $s' \in C$ then $e_{G,S}(r) = 1$ contradicting that $r \notin C_S(G)$. Hence, without loss of generality, we may assume that s' and x are in different components of $G - C$. By the fact that C is a clique, there exists a vertex $t \in C$ such that $e_{G,S}(x) \geq d_G(s', x) = d_G(s', t) + d_G(t, x) \geq d_G(s', t) + 1 \geq d_G(s', r) = e_{G,S}(r)$ contrary to that $e_{G,S}(x) < e_{G,S}(r)$. Therefore, $C_S(G)$ is m -convex. \square

Corollary 23 (Yushmanov [38,40]). *The center $C(G)$ of a chordal graph is m -convex.*

Suppose x and y are two vertices in a graph G with $d_G(x, y) = e_G(y)$. It is easily seen that if G is a tree then $e_G(x) = \text{diam}(G)$. In general, $e_G(x) \neq \text{diam}(G)$ for a distance-hereditary graph G . Moreover, the difference between $e_G(x)$ and $\text{diam}(G)$ may be arbitrarily large for a general graph. However, in the following theorem we show that $e_G(x)$ is nearly equal to $\text{diam}(G)$ for a distance-hereditary graph G . The graphs given in Fig. 3 show that the bounds in the following theorem are sharp.

Theorem 24. *For any vertex y in a distance-hereditary graph $G = (V, E)$. If x is a vertex with $d_G(y, x) = e_G(y)$, then $e_G(x) \geq \text{diam}(G) - 2$. If moreover G is Ptolemaic, then $e_G(x) \geq \text{diam}(G) - 1$.*

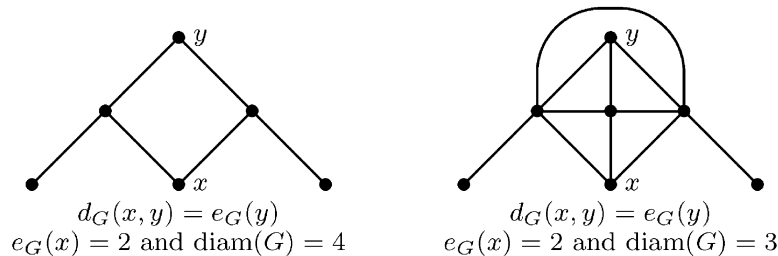


Fig. 3. Examples for which the bounds in Theorem 24 are sharp.

Proof. Let $h_u = (L_0, L_1, \dots, L_t)$ be the hanging of G at a vertex u having $e_G(u) = \text{diam}(G) = t$. Choose a vertex z in L_t . By Lemma 3, there exists a u -concave downward shortest x - y path

$$P_x : x = x_0, x_1, \dots, x_r, \dots, x_{r'}, \dots, x_k = y,$$

where x_r (respectively, $x_{r'}$) is the first (respectively, last) vertex of P_x in a smallest level L_{f_x} with $0 \leq r' - r \leq 1$; and a u -concave downward shortest z - y path

$$P_z : z = z_0, z_1, \dots, z_s, \dots, z_{s'}, \dots, z_m = y,$$

where z_s (respectively, $z_{s'}$) is the first (respectively, last) vertex of P_z in a smallest level L_{f_z} with $0 \leq s' - s \leq 1$.

We may assume that $\ell_u(z) - 2 \geq \ell_u(x)$, for otherwise $\text{diam}(G) - 1 = \ell_u(z) - 1 \leq \ell_u(x) \leq e_G(x)$ and so the theorem holds.

Suppose $x_i = z_j$ for some $i \leq r$ and $j \leq s$. Since $d_G(z_j, z) = \ell_u(z) - \ell_u(z_j) > \ell_u(x) - \ell_u(x_i) = d_G(x_i, x)$, we have $d_G(y, z) = d_G(y, z_j) + d_G(z_j, z) > d_G(y, x_i) + d_G(x_i, x) = d_G(y, x) = e_G(y)$, a contradiction. Therefore, the two paths x_0, x_1, \dots, x_r and z_0, z_1, \dots, z_s have no vertex in common.

Next, $(\ell_u(x) - f_x) + (r' - r) + (\ell_u(y) - f_x) = d_G(x, y) = e_G(y) \geq d_G(z, y) = (\ell_u(z) - f_z) + (s' - s) + (\ell_u(y) - f_z)$. Therefore, $\ell_u(z) - 2 \geq \ell_u(x)$ and $r' - r \leq 1$ and $0 \leq s' - s$ imply $f_x < f_z$. Let x_q (respectively, $x_{q'}$) be the first (respectively, last) vertex of P_x in level L_{f_x-1} . Since $x_{q'+1}$ and z_s are connected in $G - L_{f_z-1}$, by Theorem 1 (4), $x_{q'}$ is adjacent to z_s . Consider the x - z path

$$P_1 : x = x_0, x_1, x_2, \dots, x_{q'}, z_s, z_{s-1}, \dots, z_0 = z.$$

Suppose P_1 is an induced path. Note that $d_G(x, x_{q'}) + d_G(x_{q'}, y) = d_G(x, y) = e_G(y) \geq d_G(u, y) = d_G(u, x_{q'}) + d_G(x_{q'}, y)$. Then, $d_G(x, x_{q'}) \geq d_G(u, x_{q'})$ and so $e_G(x) \geq d_G(x, z) = d_G(x, x_{q'}) + d_G(x_{q'}, z) \geq d_G(u, x_{q'}) + d_G(x_{q'}, z) \geq d_G(u, z) = \text{diam}(G)$. In this case, the theorem holds.

We then may assume that P_1 is not an induced path, say P_1 has a chord joining some vertex x_i to some vertex z_j . Note that in this case $0 < q \leq r \leq r' \leq q' < k$. Then, either $i = q$ with $j = s$, or $i \leq q - 1$ with $j \leq s$. For the first case, $x_q z_s \in E$. For the second case, x_{q-1} and z_s are connected in $G - L_{f_z-1}$, and so again $x_q z_s \in E$ by Theorem 1 (4). In any case, $d_G(z_s, x_q) = 1$.

Since $d_G(y, z_s) + d_G(z_s, x_q) + d_G(x_q, x) \geq d_G(y, x) = e_G(y) \geq d_G(y, z) = d_G(y, z_s) + d_G(z_s, z)$, we have $1 + d_G(x_q, x) \geq d_G(z_s, z)$. By the fact that $d_G(x_q, x) = \ell_u(x) - \ell_u(x_q) \leq e_G(x) - (f_z - 1)$ and $d_G(z_s, z) = \text{diam}(G) - f_z$, we then have $e_G(x) \geq \text{diam}(G) - 2$. This proves the first part of the theorem.

To prove the second part of the theorem, suppose G is Ptolemaic, i.e., G is chordal and distance hereditary. For the case when $x_q = x_{q'}$, we have $q = r = r' = q'$ and $f_x = f_z - 1$. For the case when $x_q \neq x_{q'}$, since the two vertices x_q and $x_{q'}$ in L_{f_x-1} are adjacent to $z_s \in L_{f_z}$, they are also adjacent to some $w \in L_{f_z-2}$ according to Theorem 1 (4). By the chordality of G , the cycle $w, x_q, z_s, x_{q'}, w$ has a chord, which must be $x_q x_{q'}$. So, $q = r < r' = q'$ and $f_x = f_z - 1$. In any case, $d_G(x_q, x_{q'}) \leq 1$. Consider the x - z path

$$P_2 : x = x_0, x_1, \dots, x_q, z_s, z_{s-1}, \dots, z_0 = z.$$

Suppose P_2 is an induced path. Note that $d_G(x, x_q) + d_G(x_q, x_{q'}) + d_G(x_{q'}, y) = d_G(x, y) = e_G(y) \geq d_G(u, y) = d_G(u, x_{q'}) + d_G(x_{q'}, y)$ and so $d_G(x, x_q) \geq d_G(u, x_{q'}) - 1$ since $d_G(x_q, x_{q'}) \leq 1$. Therefore, $e_G(x) \geq d_G(x, z) = d_G(x, x_q) + d_G(x_q, z_s) + d_G(z_s, z) \geq d_G(u, x_{q'}) - 1 + d_G(x_{q'}, z_s) + d_G(z_s, z) \geq d_G(u, z) - 1 = \text{diam}(G) - 1$, since $d_G(x_{q'}, z_s) = d_G(x_q, z_s) = 1$. In this case, the second part of the theorem holds.

We then may assume that P_2 has a chord joining some vertex x_i to some vertex z_j with $i \leq q - 1$ and $j \leq s$. Then x_{q-1} and $z_{s'}$ are connected in $G - L_{f_z-1}$. Again, by Theorem 1 (4) and (2) and the chordality of G , $d_G(x_{q-1}, z_{s'}) \leq 1$. Thus, $d_G(y, x) \leq d_G(y, z_{s'}) + d_G(z_{s'}, x_{q-1}) + d_G(x_{q-1}, x) \leq d_G(y, z_{s'}) + 1 + d_G(x_{q-1}, x) = d_G(y, x_{q'+1}) + 1 + d_G(x_{q-1}, x) < d_G(y, x_{q'+1}) + d_G(x_{q'+1}, x_{q-1}) + d_G(x_{q-1}, x) = d_G(y, x)$, a contradiction. This completes the proof of the theorem. \square

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